PHY 456 : QM II

Sep. 11, 2023

FALL 2023
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Schedule:

(i) Elementary scattering theory.

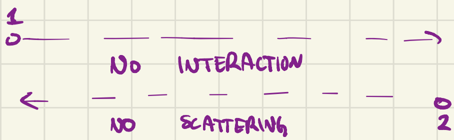
(ii) Rotations, spin and Pauli matrices, addition of $\vec{L}$

(iii) Perturbation theory.

Scattering:

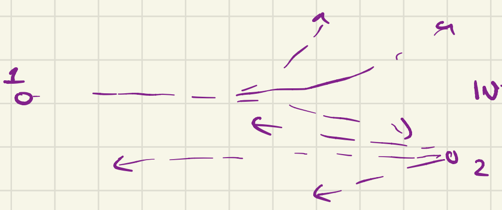
Nature of scattered stuff unchanged.

- Classical scattering:



(ELASTIC: Nature of particles $|1\rangle$, $|2\rangle$ are the same)

(inelastic: Not in 456)



INTERACTION $\Rightarrow$ SCATTERING.

- Trajectories of 1 and 2 change as a result of "interaction"
- Free particle states before & after

- Further reading:
  - L&L vol 3: QM
  - John Taylor "Scattering theory"
  - Original Rutherford paper (α-particle scattering)

- 456: NRQM, Elastic Scattering without spin/spin sums.

  ie. Particles 1, 2 interact via a potential $V(|\vec{r}_1 - \vec{r}_2|)$.

  "Hardcore"

  Coulomb, gravitational

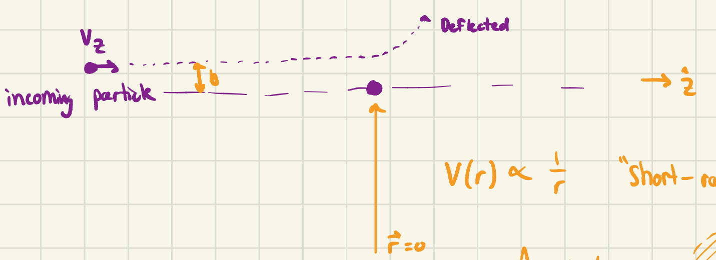
  Yukawa

Let $m_1, m_2, V(|\vec{r}_1 - \vec{r}_2|)$ define the interaction.

→ Go to CM frame. In CM frame, study motion of 1 particle (reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$) and this particle moves in $V(r)$ where $r = |\vec{r}| = |\vec{r}_1 - \vec{r}_2|$.

central pot.

- Consider now: two particles on top of each other

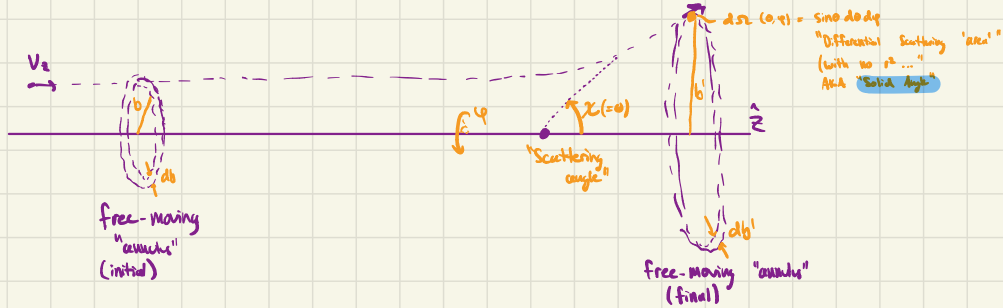


$V(r) \propto \frac{1}{r}$ "Short-ranged" $\lim_{r \to \infty} V(r) = 0$.

Approximate:

$V(r) \neq 0 \quad |\vec{r}| < |\vec{r}_0|$

$V(r) = 0 \quad |\vec{r}| > |\vec{r}_0|$



$$\begin{cases} b: \text{impact parameter} \\ \chi(=\theta): \text{scattering angle, a function of } \chi(b, E = \frac{1}{2}mv_z^2) \end{cases}$$

↳ (Find χ by solving CM problem)

- Imagine we have a beam of particles moving along the $\hat{z}$ axis with some v_z's and b's.

flux I (intensity) = # incoming particles passing a unit area $\perp$ to $\hat{z}$ axis with unit time, all same E or v_z.

$dN(\theta,\varphi)$ = # particles deflected in $d\Omega(\theta,\varphi)$ far away from $r=0$ in unit time.

Intuitively, $dN(\theta,\varphi) \propto I$ (more particles = more scattering).

Thus $dN(\theta,\varphi) = d\sigma(\theta,\varphi)\, I$ where $d\sigma(\theta,\varphi)$ is the differential scattering cross section.

Also, $\dfrac{d\sigma(\theta,\varphi)}{d\Omega(\theta,\varphi)}$ is also called the differential scattering cross-section.

↳ has same units as $d\sigma(\theta,\varphi)$, since solid angle is dimensionless.

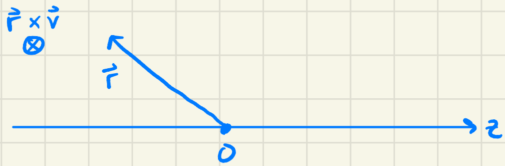
We find that

depends only on energy $\rightarrow \dfrac{d\sigma(\theta,\varphi)}{d\Omega(\theta,\varphi)} = \dfrac{dN(\theta,\varphi)}{I \; d\Omega(\theta,\varphi)}$ $\rightarrow$ has dimensions of area L^2

$$\left[\frac{\frac{\#}{t}}{\frac{\#}{A}\;\frac{1}{t}}\right] \rightarrow A = L^2.$$

Classical Intuition: we have $E = \frac{1}{2} m v_\infty^2$ where $V_\infty = V_z \,(z \to -\infty)$.

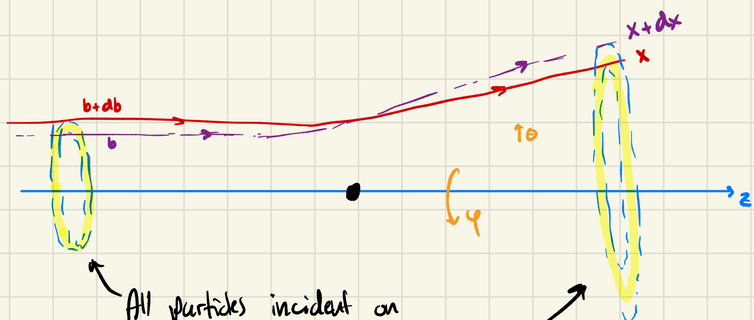
Angular momentum $L = m V_\infty b$



As before, solving trajectories means finding

$\chi(b, V_\infty)$

$\chi = \theta \rightarrow$ Assume central potential, motion is planar



All particles incident on this ring end up in this ring

(Azimuthal symmetry assumed)

We also assume a steady state process, ie $\dfrac{dN}{dt} = 0$

(# particles conserved) and follow deterministic trajectories.

observed in scattered ring $\chi, \chi + d\chi$.

Hence $dN = \underbrace{2\pi I b \, db}_{\substack{\#\text{ particles in } b,\, b+db \\ \text{end up in } \chi,\, \chi + d\chi}}$.

We know that $\chi = \chi(b)$, so $b = b(\chi)$

(we assume for simplicity that a particle scattered at χ comes from a given b)

Thus: $dN = 2\pi b I \, db$

$$= 2\pi I \, b(\chi) \, db(\chi)$$

which immediately implies

$$\Longrightarrow d\sigma(\chi) = 2\pi \, b(\chi) \, db(\chi)$$

$$= 2\pi \, b(\chi) \left| \frac{db(\chi)}{d\chi} \right| d\chi$$

sign doesn't matter (usually $b'(\chi) < 0$ because smaller b means bigger χ since you're closer to the potential / atom)

$(\chi \equiv \theta)$

or $d\sigma(\chi) = \frac{b(\chi)}{\sin\chi} \left| \frac{db(\chi)}{d\chi} \right| \underbrace{2\pi \sin\chi \, d\chi}_{d\Omega}$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{b(\chi)}{\sin\chi} \left| \frac{db(\chi)}{d\chi} \right|$$

(CM problem: solve $\chi(b)$, $b(\chi)$)

$\equiv$ differential cross-section.

This characterizes scattering, beam intensity factored out; depends on $V(r)$ and x $(x=\theta)$ for central potentials.

Use?

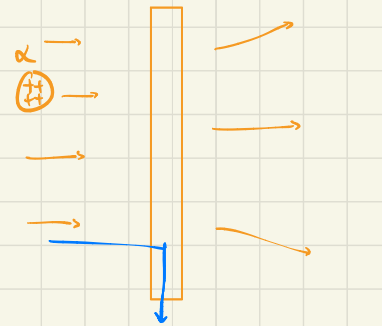
$$d\Omega \cdot \frac{d\sigma}{d\Omega} \cdot I \cdot T = \#\text{ scattering events in } d\Omega .$$

I: Intensity (Flux)

T: Length of experiment

But is this theory correct?

- **Rutherford 1909-1911:** Gold foil.



$$\frac{d\sigma}{d\Omega} = \left(\frac{z z' e^2}{4E}\right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}$$

$z' = 4$ (α-particle)

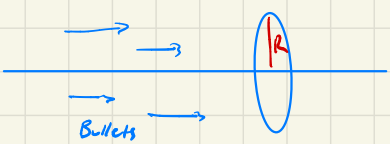
z = nucleus charge

- Small fraction scatter a lot!
- Measure as function of E; distance of closest approach $\sim \gtrsim$ size of nucleus.

A famous formula in C-mech and Q-mech (Born approximation) and QED.

Last few comments before QM:

1) HARD DISK:



Then $\sigma = \pi R^2$

$$\left(\frac{\#\text{ scattered}}{\text{flux}} = \text{area of disk}\right)$$

2) FOR RUTHERFORD:

$$\int d\Omega \frac{d\sigma}{d\Omega} \to \infty \quad \Rightarrow \quad \int_0^{\pi} dx \frac{\sin x}{\sin^2(\frac{x}{2})} \to \infty,$$

which means that in Coulomb potentials, all particles scatter, even as $b \to \infty$ (not finite range!)

QM: use SE (not trajectories) to describe scattering.

• <u>Quantum Mechanics</u>:

- Use steady-state description of scattering
- Let ①, ② interact with $V(|\vec{r}_1 - \vec{r}_2|)$.
- Separate CM-motion, consider a single particle of mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ in $V(|\vec{r}|)$

$$H = \frac{\vec{p}^2}{2m} + V(\vec{r}), \quad \text{let} \quad \Psi(\vec{r}, t) = e^{-iEt/\hbar} \varphi(\vec{r})$$

where $\varphi(\vec{r})$ satisfies $\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\varphi = E\varphi$.

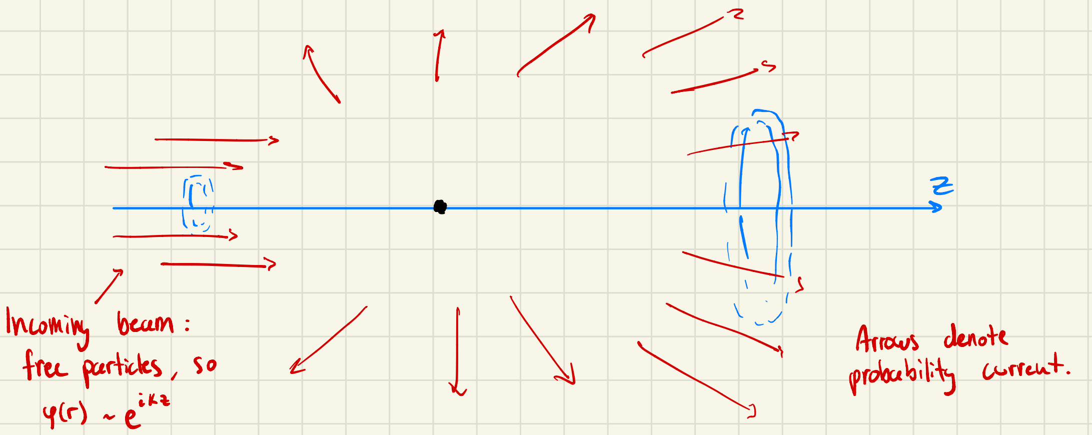
Use standard notation $E = \frac{\hbar^2 k^2}{2\mu} \geq 0$.

We want states describing particle motion $r \gg r_0 = 0$, so $E > 0$ and not bound states:

$$u(\vec{r}) = \frac{2\mu}{\hbar^2} V(\vec{r}) \quad \text{and}$$

$$\left(\nabla^2 + k^2 - u(\vec{r})\right) \varphi(\vec{r}) = 0.$$

a steady state process should be described by a stationary solution to the Schrödinger equation.



We therefore want scattering solutions far from $\vec{r}_0 = \vec{0}$.

$\left(\nabla^2 + k^2 - u(\vec{r})\right) \varphi(\vec{r}) = 0$ therefore solves our equation for $|\vec{r}| \to \infty$, since $u(\vec{r})$ is assumed to be finite range.

Scattered component: at $r \to \infty$ $\left(\nabla^2 + k^2\right) \varphi(\vec{r}) = 0$

so $\varphi = e^{ikz}$ solves.

Other solutions: Consider radial-only components

$$\nabla^2 \equiv \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d}{dr}\right) \qquad \text{(fix } \theta, \varphi\text{)}$$

Claim: $$\left(\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d}{dr}\right) + k^2\right) \frac{e^{ikr}}{r} = 0 \Bigg|_{r > \varepsilon} \quad (\varepsilon > 0).$$

$$(\text{or}) \quad \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2\right)\frac{e^{ikr}}{r} = 0.$$

acting 2^x only e^{ikr} cancels k^2, drop. Rest is:

We have that:

$$= \cancel{e^{-ikr} \cdot 2 \frac{d}{dr} e^{ikr}} \cdot \frac{d}{dr}\frac{1}{r} + \frac{d^2}{dr^2}\frac{1}{r} + \frac{2}{r}\frac{d}{dr}\frac{1}{r} + \cancel{\frac{2}{r^2} e^{-ikr}\frac{d}{dr}e^{ikr}}$$

$$= \frac{2}{r^3} - \frac{2}{r^3}$$

$$= 0.$$

Therefore it would be that

outgoing spherical wave

(k - energy, r - radius from $V(\vec{r})$)

$$\varphi(\vec{r})\Big|_{r\to\infty} = \underbrace{e^{ikz}}_{\text{incoming plane wave}} + \underbrace{f_k(\theta,\varphi)}_{\equiv \text{"Scattering Amplitude"}} \frac{e^{ikr}}{r}$$

which is what we want. Recall our definitions:

$$E = \frac{\hbar^2 k^2}{2\mu}$$

$$\left(\nabla^2 + k^2 - u(\vec{r})\right)\varphi(\vec{r}) = 0$$

$$u(\vec{r}) = \frac{2\mu}{\hbar^2} V(\vec{r})$$

$\therefore$ Our $\varphi(\vec{r})$, with asymptotics, is called the scattering solution.

How is φ^{SCATT} related to $d\sigma(\theta, \varphi)$?

Use probability currents associated with φ^{SCATT} !

Recall: $\rho(\vec{r}, t) \equiv \psi^* \psi \, (\vec{r}, t)$ with $\hat{H}\psi = E\psi$.

Let $\vec{j}(\vec{r}, t) \equiv -i \frac{\hbar}{2\mu} \left(\psi^* \nabla \psi - \psi \nabla \psi^* \right)$ = "probability current".

Then SE implies probability conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0.$$

In other words,

$$\frac{d}{dt} \int_V d^3r \, \rho(\vec{r}, t) = - \oint_{\partial V} d^2\vec{s} \cdot \vec{j}$$

(See tut or notes).